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CONVOLUTION SPLINES BASE OF AN ARBITRARY CLASS OF DIFFERENTIABILITY

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Abstract

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1. INTRODUCTION

The basic tool is a partition of unity obtained through *regularization by convolution* of the characteristics of a partitioning of the interval $[A, B]$, using *kernel with compact support*. as has been done for example in [6].

1.1. *Analysis of the computational algorithm*

f is the nth power by convolution of $\chi_{[0,1]}$, here $n = 5$.

Here *kernel* as fixed element of a sequence that is an *approximate unit*, [9, ch 6, page 157] used as a convolution factor for regularization. Any convolution power of $\chi_{[0.1]}, f$ is a kernel and in [5], we have established a algorithm to construct f for whatever n, a specific power will be pointed by the order foa differential operator under study. In this paper, $[0, m] = [A, B]$, where m is an integer representing the number of integer nodes of the partition. The transform equations of f for $[A, B]$ are

$$\eta(x) = nf(nx); \rho(x) = \eta(x + \frac{1}{2}); \rho_k = k\rho(kx); \qquad (1)$$

$$R = B - A; w = \frac{m}{R}; \eta_w = w\eta(wx); \qquad (2)$$

$$a(x) = \eta * \chi_{[0,1]}; a_w(x) = \eta_w * \chi_{[0,\frac{1}{w}]}; \qquad (3)$$

$$supp(a_w) = [0, \frac{2}{w}]; \rho_{w,k} = k\rho_w(kx); \qquad (4)$$

and $\rho_k = k\rho(kx)$ will be kernel we shall use.

A `python` class was built to represent the mathematical objects of this paper, to be soon published. We have used Riemann sums to have the integral of $h(x) = x^2$ on $[-1, 1]$ precise to 5th figure, integral of the kernel and its derivative precise up to 15th figure. Compared cosine and its convolution approximation at the $x = 3, 4, 7$ precise to 2nd figure. The Figure (1) at page 1, shows the graphics of $h(x) = sin(3x + 7)(\frac{x}{2})^2$ and $h * \rho_{10}; \rho_{10} : x \mapsto 10\rho(10x)$; The graphics are overshadowing one another, $\|h(x) - (h * \rho_{10})(x)\| \leq 0.036; x \in [-10, 14]$ step $delta = 0.2$.

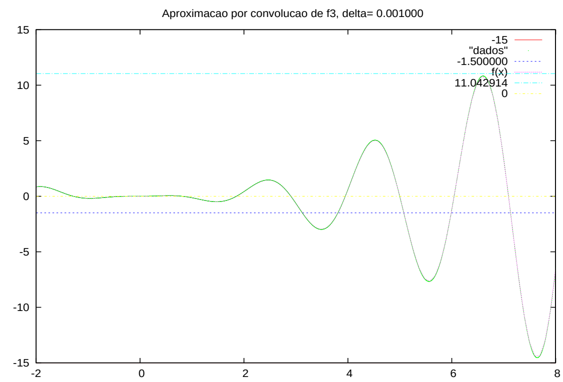

Figure 1 – convol. approx. of $h(x) = sin(3x + 7)(\frac{x}{2})^2$

2. CONSTRUCTION OF A PARTITION OF THE UNITY

The calculations sketch how to produce a splines partition of the unity:

$$f(x) = \chi^n_{[0,1]}; \eta(x) = nf(nx); m(supp(\eta)) = 1; \qquad (5)$$

$$h_j(x) = h(x - j) = (h * \delta_j)(x); \qquad (6)$$

$$\sum_{j=0}^{m} \chi_{[0,1]}(x - j) = 1 \text{ q.s.} \qquad (7)$$

$$a = \eta * \chi_{[0,1]}; a_j(x) = a * \delta_j; \qquad (8)$$

$$\eta * (\sum_{j=0}^{m} \chi_{[0,1]}(x - j)) = 1; x \in \mathcal{V}_{[1,m]} \qquad (9)$$

$$(a_j)_{j\in\{0,...,m\}} = (a * \delta_i)_{i\in\{0,...,m\}}; \qquad (10)$$

$supp(a)$ is an interval of measure 2, the successive $j-$translations coincides exactly over the half-interval, this makes the point for the optimization of the algorithm:

$$[0, m] \ni x \mapsto \sum_{j=0}^{m} a_j(x) = a_k(x) + a_{k+1}(x); k = \lceil x \rceil \qquad (11)$$

Convolutions replaced by translations of the convolution already done, [5] the elements at Eq. (11), the partition relative to $[0, m]; m > 0; m \in \mathbf{N}$, but equations (4) - (7), will reset to the general case, power n, $[A, B]$, m nodes, a program to be published, shortly, under GPL does this.

3. INTERPOLATION PROJECTOR

To build the interpolation projector associated with a partition the interval $[0, m]$ we use the system of equations (5)-(10).

3.1. *Projector* $PU(g)$

Method: a trivial example generalized later. Graphic in Figure (2), page 2, is the linear interpolation of selected

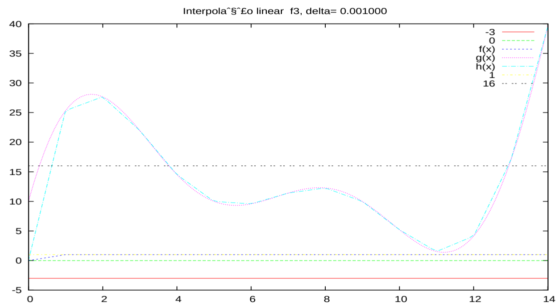

Figure 2 – Interpolation projector: linear interpolation of $g(x) = sin(x/2)(x-5)(x-9)+10)$

points on the graph of g

The atom $a(x)$, Eq. (8) has support $[0, 2]$ at which translations have the peeking point of $a_j(x)$ to coincide with the minimum of $a_{j+1}(x)$, see Figure (3). The sum $\sum_{j=0}^{m} a_j(x)$ will

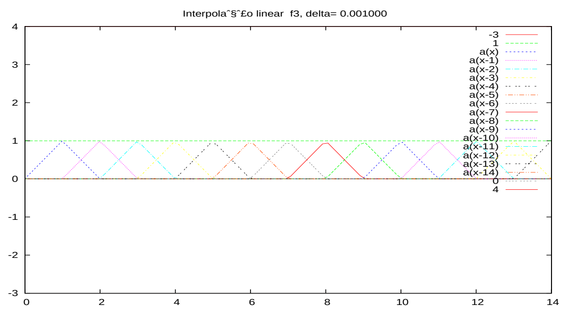

Figure 3 - integer translations of an atom of compact support

be a straight line at the Figure (3). Linear interpolation is the sum of the translations using coefficients $g(i)$ to $a(x-i)$, the integers nodes: the precision points. An other explanation, $h(x) = \sum_{j=0}^{m} a_j(x)$ is the sum of the convolution products of $\chi_{[j,j+1];j=0\ldots m}$ with $\chi_{[0,1]}$. Put $\eta(x) = nf(nx)$ instead of $\eta = \chi_{[0,1]}$ to have a partition of unity of class $\mathcal{C}^{n-2}$.

But the point here is the sum

$$\sum_{j=0}^{m} g(j)a_j \tag{12}$$

that leads to the expression of interpolation project we are searching for:

Definition 1 (Projector) *interpolation projector*

Consider the vector space $\mathcal{SP}_m([0, m])$ *generated by translations of atoms* a *defined in equation (8).*

$$\sum_{j=0}^{m} g(j)a_j \tag{13}$$

defines an interpolation projector of the space of continuous functions $\mathcal{C}([a, b], \mathbf{R})$ *into the space* $\mathcal{SP}_m([A.B])$.

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